**Time Value of Money**

**Learning Outcomes**

After completing this module, students will be able to:

1. Compute the present and future values of varying cash payment patterns including single payments, annuities, annuities due, annuities with growth, and perpetuities with growth.
2. Explain the concepts of simple and compound interest and continuous compounding.
3. Demonstrate the difference between annual percentage rates and effective annual rates.
4. Calculate payments under different loan repayment options including discount, interest-only, and full and partial amortization.
5. Derive the different present and future value formulas.
6. Solve future and present value problems using predefined financial functions in Excel.

**Introduction**

Would a manager rather receive payment from a customer today or in one year? Being paid today is better as that money could be invested at current interest rates generating a larger amount in a year. Would a manager rather pay a supplier now or in one year? Paying later is preferable as a lesser amount could be invested now so that in a year with interest the company would have enough to pay the supplier. These are examples of the time value of money.

Frequently managers must determine what a single cash payment or series of cash payments is worth either today or at a future date. Being able to accurately value the cash inflows and outflows that a business or one of its projects generates is critical in making effective management decisions. Not correctly incorporating the time value of money can lead to critical errors.

* 1. **| Future Value and Compounding**

The future value (FV) of an investment equals the initial principal plus accumulated interest at a specific point in time. After a single period, investors will receive their initial principal as indicated by the first component of the formula below plus interest equal to the principal (P) times the interest rate (i).

$FV=\left(P\right)\left(1\right)+\left(P\right)\left(i\right)$ or $FV=\left(P\right) \left(1+i\right)$

Over multiple periods (n), FV depends on whether the interest is paid out or reinvested. If it is re-invested, it is added to the initial principal each period, and the interest earned in subsequent periods increases as the principal grows. After three periods, the FV would be calculated as:

$$FV=\left(P\right) \left(1+i\right) \left(1+i\right) \left(1+i\right)$$

This is called compound interest and investors are said to be earning interest on their interest. The formula can be simplified, so the FV equals the initial principal times the FV factor.

FV factor

$$FV= \left(P\right)\left(1+i\right)^{n}$$

FV factor tables were published for different interest rate (i) and period (n) combinations before calculators and microcomputers become available in the 1970s and 1980s, but they are used sparingly today. Electronic calculating devices have predefined financial functions that determine FV automatically.

**Exhibit 1: Future Value with Compound Interest**

|  |  |  |  |
| --- | --- | --- | --- |
| **Year**  | **Beginning Principal (CAD)** | **Compound Interest****i = 5.0%** | **Ending Principal (CAD)** |
| 1 | 1,000.00 | (1 + 0.05) = 1.05 | 1,050.00 |
| 2 | 1,050.00 | (1 + 0.05) = 1.05 | 1,102.50 |
| 3 | 1,102.50 | (1 + 0.05) = 1.05 | 1,157.63 |
| **FV factor = (1 + 0.05) (1 + 0.05) (1 + 0.05) = (1 + .05)3= 1.15763** |

If interest is paid out, the initial principal does not change so the interest earned in future periods will be the same. This is called simple interest and is lower than compound interest over time due to the lack of compounding.

**Exhibit 2: Simple Versus Compound Interest**

|  |  |  |
| --- | --- | --- |
| **Year** | **Simple Interest** | **Compound Interest** |
| **Beginning Principal (CAD)** | **Interest****i = 5.0%** | **Beginning Principal (CAD)** | **Interest****i = 5.0%** |
| 1 | 1,000.00 | 50.00 | 1,000.00 | 50.00 |
| 2 | 1,000.00 | 50.00 | 1,050.00 | 52.50 |
| 3 | 1,000.00 | 50.00 | 1,102.50 | 55.13 |
|  | **Total interest** | **150.00** | **Total interest** | **157.63** |
| **Extra interest due to compounding = 157.63 -150.00 = CAD 7.63** |

As shown in the exhibit below, disciplined investors who re-invest their interest instead of spending the cash can greatly increase their return especially as interest rates and the length or term of the investment rise. This effect is sometimes called the “magic” of compound interest and is an important concept for long-term investors such as pension plans or life insurance companies.

**Exhibit 3: Effect of Compound Interest Over Time**

Sometimes a manager knows what FV they need from an investment but may have to adjust the initial principal (P), interest rate (i), or the number of periods (n) to reach that target. The FV formula can be used to conduct a “what if” analysis to determine the appropriate P, i, or n.

**Compounding Period**

Interest rates are quoted annually for comparability, but the compounding period can be annually, semi-annually, quarterly, monthly, or daily. To correctly incorporate compound interest in the FV formula, the interest rate (i) must be per compounding period and the number of periods (n) must be the number of compounding periods over the life of the investment. To determine the interest rate (i) per compounding period, the quoted annual interest rate is divided by the number of compounding periods per year (i.e. 365 days, 12 months, 4 quarters, or 2 six-month periods, or 1 year). The number of compounding periods is the number of compounding periods per year times the life of the investment in years. For example, a two-year investment with a quoted annual interest rate of 5.0%, compounded quarterly, would have an interest rate of 1.25% (i.e. 5.0% ÷ 4 quarters per year) per compounding period and eight compounding periods (i.e. 4 quarters per year x 2 years).

The quoted annual interest rate is referred to as the annual percentage rate (APR), but if the compounding period is more frequent than annually then the effective annual rate (EAR) will be higher. More frequent compounding gives investors a better return or is more costly to borrowers. The EAR is calculated as:

$$EAR=\left(1+ \frac{APR}{Number of compounding periods per year}\right)^{Number of compounding periods per year}-1$$

EAR provides firms with their true return or cost of borrowing per annum.

Interest rates can also be compounded continuously which assumes an infinite number of compounding periods that results in the highest possible EAR. It is not used in commercial lending, but EAR does have some applications in finance such as derivative valuation.

EAR = $e^{APR}-1$

EAR with continuous compounding is a mathematical limit equal to the natural logarithm of APR to the base of the constant *e* which is approximately 2.71828.

**Exhibit 4: APR versus EAR**

|  |  |  |
| --- | --- | --- |
| **APR** | **Calculation** | **EAR** |
| 5%, compounded annually | ($1+ \frac{.05}{1}$)1 − 1 = .0500 | 5.00% |
| 5%, compounded semi-annually | ($1+ \frac{.05}{2}$)2 − 1 = .0506 | 5.06% |
| 5%, compounded quarterly | ($1+ \frac{.05}{4}$)4 − 1 = .0509 | 5.09% |
| 5%, compounded monthly | ($1+ \frac{.05}{12}$)12 − 1 = .0512 | 5.12% |
| 5%, compounded continuously | $$2.71828^{.05}-1= .0513$$ | 5.13% |

**1.2 | Present Value and Discounting**

Present value (PV) is closely related to the concept of FV. In the last section, formula 1 below was used to determine the initial principal required to achieve a certain FV assuming a specific interest rate and term. The initial principal is also called the PV as it is expressed in today’s dollars or, as finance people say, at time (T) = 0. PV is substituted for P and then formula 1 is rearrange giving formula 2. PV equals FV divided by the PV or discount factor.

 $Formula 1 FV= \left(P\right)\left(1+i\right)^{n}$

$$Formula 2 PV=\frac{FV}{\left(1+i\right)^{n}}$$

PV or discount factor

Managers often undertake capital projects where they make a large initial cash investment in land, plant, and equipment. The project generates positive cash flows in the future, but managers need to know if these future benefits are greater than the initial costs. The problem is the cash outflows and inflows do not occur at the same time. To make them comparable, the PV of the future cash flows are determined so they can be compared to the initial cash flows which are already at T= 0. If the PV of the future cash inflows exceeds the initial cash outflows, the project is undertaken.

**Exhibit 5: Evaluating a Capital Project**

|  |  |  |  |
| --- | --- | --- | --- |
| **Time** | **Cash Flows (CAD)** | **Calculation****i = 8.0%, compounded annually** | **Present Value (CAD)** |
| Cash outflows | -150,000 | $$= \frac{-150,000}{\left(1+ .08\right)^{0}}$$ | -150,000 |
| Cash inflows at end of year 1 | 75,000 | $$= \frac{75,000}{\left(1+ .08\right)^{1}}$$ | 69,444 |
| Cash inflows at end of year 2 | 100,000 | $$= \frac{100,000}{\left(1+ .08\right)^{2}}$$ | 85,734 |
| Cash inflows at end of year 3 | 120,000 | $$= \frac{120,000}{\left(1+ .08\right)^{3}}$$ | 95,260 |
| **Total** | **CAD 100,438** |

Like with the FV formula, managers can adjust FV, interest rate (i), or the number of periods (n) in the PV formula as part of a “what if” analysis to determine the actions needed to achieve a certain PV.

**1.3 | Annuities and Perpetuities**

**Annuities**

Finite series of equal payments are common in business with different cost or revenue streams such as loan payments or investment returns. If the payments occur at the end of each period, they are called an ordinary annuity or just an annuity. If they occur at the beginning of each period, they are called an annuity due.

**Exhibit 6: Future Value of an Annuity or Annuity Due**

|  |  |
| --- | --- |
| **Annuity (CAD)****i = 5.0%, compounded annually** | **Annuity Due (CAD)****i = 5.0%, compounded annually** |
| **Payment** | **FV Factor** | **FV** | **Payment** | **FV Factor** | **FV** |
| 5,000 | (1 + .05)2 | 5,512.50 | 5,000 | (1 + .05)3 | 5,788.12 |
| 5,000 | (1 + .05)1 | 5,250.00 | 5,000 | (1 + .05)2 | 5,512.50 |
| 5,000 | (1 + .05)0 | 5,000.00 | 5,000 | (1 + .05)1 | 5,250.00 |
| **Total** | **15,762.50** | **Total** | **16,550.62** |
| **Note:** The exponents vary depending on whether the payments occur at the beginning (ordinary annuity) or ending (annuity) of the year. |

**Exhibit 7: Present Value of an Annuity or Annuity Due**

|  |  |
| --- | --- |
| **Annuity (CAD)** **i = 5.0%, compounded annually** | **Annuity Due (CAD)** **i = 5.0%, compounded annually** |
| **Payment** | **PV Factor** | **PV** | **Payment** | **PV Factor** | **PV** |
| 5,000 | (1 + .05)1 | 4,761.90 | 5,000 | (1 + .05)0 | 5,000.00 |
| 5,000 | (1 + .05)2 | 4,535.15 | 5,000 | (1 + .05)1 | 4,761.90 |
| 5,000 | (1 + .05)3 | 4,319.19 | 5,000 | (1 + .05)2 | 4,535.15 |
| **Total** | **13,616.24** | **Total** | **14,297.05** |

The PV or FV of each annuity payment can be calculated individually and summed to give the value of the annuity or formulas can be used.

|  |  |
| --- | --- |
| Present value of an annuity (PVA) | P ($\frac{1-\left(1+i\right)^{-n}}{i}$) |
| Present value of an annuity due (PVAD) | P ($\frac{1-\left(1+i\right)^{-n}}{I} ) $(1 + i) |
| Future value of an annuity (FVA) | P ($\frac{\left(1+i\right)^{n}-1}{i}$) |
| Future value of an annuity due (FVAD) | P ($\frac{\left(1+i\right)^{n}-1}{i}) $(1 + i) |

**Perpetuities**

An infinite series of equal payments is called a perpetuity. Perpetuities are less common than annuities in business but there are examples such as common or preferred share dividends which are paid out regularly forever. Perpetual bonds also make fixed interest payments each period, but the bonds never mature.

Trying to determine the FV of a perpetuity is illogical as the series never ends. The PV of a perpetuity (PVP) can be calculated because the PV of future payments eventually becomes so small because of the long discounting period that the PVP reaches a mathematical limit which means it does not rise past a certain point.

|  |  |
| --- | --- |
| Present value of a perpetuity (PVP) | PVP = $\frac{P}{i}$ |

* 1. **| Growing Annuities or Perpetuities**

Some business applications have a finite or infinite series of payments, but they are not equal as they grow at a constant rate due to inflation or real growth. The formulas for the PV of an annuity or perpetuity with growth are:

|  |  |
| --- | --- |
| Present value of an annuity with growth (PVAG) | $\left(\frac{P}{i - g}\right)$ (1 – $\left( \frac{1+g}{1+i}\right)^{n}$) |

|  |  |
| --- | --- |
| Present value of a perpetuity with growth (PVPG) |  $\frac{P}{i – g}$ |

To use either of these formulas, the growth rate (g) must be constant and be less than the interest rate (i) as a negative number in the denominator is illogical. This may be the case for mature companies, but for high-growth companies, it may be necessary to use a two or three-stage model with a higher growth rate for the initial period(s) before the growth rate falls to a long-term level that is less than the interest rate. During the high-growth stages when the growth rate exceeds interest, the PV of each cash flow is calculated separately for each period. For subsequent periods, the PVP with growth formula can be used. This approach is commonly used when valuing common shares.

**1.5 | Commercial Lending**

Commercial lending is an important application of the time value of money concept, especially present value. Businesses that borrow funds must pay interest and repay the principal in the future. The three basic types of loans are discount, interest-only, and amortized.

**Discount loans.** For short-term loans under a year, the borrower typically receives a fraction of the loan’s face value today but then pays back the full amount at maturity. The difference between the two amounts is the interest expense. Issuing loans this way is administratively more convenient for the lender as they do not have to calculate and collect multiple interest and principal payments. For example, a business agreed to repay CAD 25,000 in one year when a loan matures. If the loan has an interest rate of 5.5%, compounded monthly, the borrower would receive CAD 23,665.10 today. The extra CAD 1,334.90 paid at maturity is the interest on the loan.

PV = $\frac{FV}{\left(1+i\right)^{n}}$= $\frac{25,000}{(1+\frac{.055}{12})^{12}} $= 23,655.10

**Interest-only loans**. For long-term loans, interest is paid regularly on the outstanding balance, but principal repayment procedures vary. Interest-only (IO) loans stipulate that the borrower pays interest on the principal during the life of the loan and repays the principal when the loan matures. IO loans are risky for lenders as most loans are backed up by collateral. Collateral is property pledged by the borrower that the lender can take if the borrower fails to make their payments. The collateral is typically the assets purchased with the borrowed funds such as new equipment. These assets can depreciate quickly, so it is important to start repaying the principal right away to ensure that the value of the collateral remains safely above the balance of the loan. Only more creditworthy borrowers will likely qualify for IO loans as they have strong operating cash flows and other unpledged collateral that can be used to service the loan.

**Amortized loan.** These loans are normally repaid in blended, equal monthly payments of interest and principal over the amortization period or life of the loan. Some amortized loans repay principal on a straight-line basis with interest paid on the remaining balance. Other loans may not require principal repayment initially as the project or company becomes established, but then have increased or “stepped” principal payments in subsequent periods as cash flows increase with potentially a large “balloon” or “bullet” payment at maturity.

Amortized loans being repaid in blended, equal monthly payments of interest and principal are annuities. When the PVA is calculated using the current market interest rate, it removes the interest component of each payment. What remains is the initial principal of the loan.

Initial Loan Principal = Payment ($\frac{\left(1 – \left(1+i\right)^{-n}\right)}{i}$)

Using the principal, monthly interest rate, and monthly payment, an amortization table can be prepared with the beginning and ending principal for each period along with the interest and principal component of each payment. As the amortization period is extended, the payments become smaller and the principal component of each payment falls relative to the interest component. Borrowers will benefit from smaller payments but will pay more interest over the life of the loan. The remaining principal is always zero by the end of the amortization period.

**Exhibit 8: Amortization Table with Blended, Equal Monthly Payments**

|  |  |
| --- | --- |
| **Amount of loan:** CAD 100,000  | **Interest rate:** 6%, compounded monthly |
| **Amortization period:** 25 years  | **Monthly payment:1** CAD 644.30 |
|  **Period** | **Beginning Principal** | **Interest****(.005)** | **Principal** | **Ending Principal** |
| 1 | 100,000.00 | 500.00 | 144.30 | 99,855.70 |
| 2 | 99,855.70 | 499.28 | 145.02 | 99,710.68 |
| 3 | 99,710.68 | 498.55 | 145.75 | 99,564.93 |
| Periods 4 through 299 |
| 300 | 642.06 | 3.21 | 641.09 | 0.97 |
| 1100,000 = P$ (\frac{1-\left(1+ \frac{.06}{12}\right)^{-(25 x 12)}}{\frac{.06}{12}})$ P = 644.30 |

**Goal Seek**

When conducting a “what-if” analysis with the more complicated annuity and perpetuity formulas, it is sometimes difficult to isolate for interest (i). Goal Seek is a tool in Excel that allows users to solve for one unknown that appears in multiple places in an equation. For example:

PVA = P ($\frac{\left(1+i\right)^{n}-1}{i}$)

Goal Seek can be found in Excel by selecting Data across the top menu and then What-if Analysis. The dialogue box for Goal Seek prompts the user for three inputs:

**Exhibit 9: Using Goal Seek**

|  |  |
| --- | --- |
| Set Cell | Indicate the cell location of the formula that incorporates all variables except the unknown variable interest rate (i). Only enter the equation to the right of the equal sign. Do not enter “i” as Excel does not recognize this character but instead use a cell address such as A1. |
| To Value | Excel automatically substitutes interest rates (i) in the formula until it equals this value which is the PVA. The actual value must be entered into Goal Seek, so a cell address cannot be used. |
| By Changing Cell | The cell contains the interest (i) that equates the Set Cell and To Value, but it is per compounding period, so it needs to be converted into an APR or EAR. Enter a number in A1 before selecting OK to start Goal Seek otherwise, Excel will not be able to determine if A1 is a letter or a number. |

Commercial lending is examined in greater detail in Module: Maturity Matching and Module: Permanent Debt and Equity Financing.

**1.6 | Present and Future Value Formulas**

Understanding how the different present and future value formulas are derived mathematically enhances a user’s ability to apply the time value of money in different business situations.

**Exhibit 10: Proofs for PV and FV Formulas**

|  |  |
| --- | --- |
| **Formula** | **Explanation** |
| PVP |  $\frac{P}{i}$ | The detailed formula for a PVP is:PVP = $\frac{P }{\left(1+i\right)^{1}}+ \frac{P}{\left(1+i\right)^{2}}+…$ + $\frac{P}{\left(1+i\right)^{\infty }}$Both sides of the formula are multiplied by (1+i) and then simplified:(1+i) (PVP) = $P + \frac{P}{\left(1+i\right)^{1}}+…$ + $\frac{P}{\left(1+i\right)^{\infty }}$PVP + (i) (PVP) = P + PVP(i) (PVP) = PPVP = $\frac{P}{i}$ |
| PVA | P ($\frac{1-\left(1+i\right)^{-n}}{i}$) | PVP beginning in n periods is discounted for n periods and subtracted from a PVP beginning today. The difference is a PVA for n periods beginning today.$\frac{P}{i}-\left(\frac{P }{i}\right)\left(\frac{1}{\left(1+i\right)^{n}}\right)$ = P ($\frac{1-\left(1+i\right)^{-n}}{i}$) |
| PVAD | P ($\frac{1-\left(1+i\right)^{-n}}{i}$)(1 + i) | PVA is multiplied by (1+i) since the payments are at the beginning of the period with an annuity due, so they were discounted for one too many periods. An alternative formula is:P + P ($\frac{1-\left(1+i\right)^{-\left(n-1\right)}}{i}$)The first payment does not have to be discounted as it occurs at the beginning of the period. Only the remaining payments (n−1) are discounted. A payment occurring at the beginning of the second period is equivalent to one occurring at the end of the first period in terms of time so the PVA formula can be used. |
| FVA | P ($\frac{\left(1+i\right)^{n}-1}{i}$) | FVA is calculated by determining the PVA, finding the future value of this single amount by compounding it for the life of the annuity, and then simplifying the formula. P ($\frac{1-\left(1+i\right)^{-n}}{i}$)$\left(1+i\right)^{n}$ = P ($\frac{\left(1+i\right)^{n}-1}{i}$) |
| FVAD | P ($\frac{\left(1+i\right)^{n}-1}{i}$) (1 + i) | FVA is multiplied by (1+i) since the payments occur at the beginning of the period with an annuity due, so one additional compounding period is required. |
| PVPG | $$\frac{P}{i - g}$$ | The formula for PVP is:PVP = $\frac{P }{\left(1+i\right)}+ \frac{P \left(1+g\right)}{\left(1+i\right)^{2}}+… $+ $\frac{P}{\left(1+i\right)^{\infty }}$This is an infinite geometric series that has the standard formula:PVP = a (1 + x + x2+ …), where a = $\frac{P}{\left(1+i\right)}$ and x = $\frac{\left(1+g\right)}{\left(1+i\right)} .$The sum of an infinite geometric series is PVP = $\frac{a}{1-x}$ .Substituting a and x and simplifying, PVP =$ \frac{P}{i – g}$ . |
| PVAG |  $\frac{P}{i – g}$ (1 – $\left( \frac{1+g}{1+i}\right)^{n}$) | PVPG beginning today minus the PVPG in n periods discounted to today equals the PVAG for n periods beginning today.  $\frac{P}{i - g}-\left(\frac{P \left(1+g\right)^{n}}{i - g}\right)\left(\frac{1}{\left(1+i\right)^{n}}\right)$ = $\frac{P}{i – g}$ (1 – $\left( \frac{1+g}{1+i}\right)^{n}$) |

**1.7 | Predefined Future Value and Present Value Functions in Excel**

Financial calculators and spreadsheet software such as Excel include predefined financial functions (*fx*) that perform many of the PV and FV operations examined in this module. Studying the mathematical formulas gives analysts a better understanding of the time value of money concept, but the functions can save time. The exhibit below contains some of the financial functions available in Excel. Its Help feature provides a more thorough explanation.

**Exhibit 11: Predefined Functions in Excel**

|  |  |
| --- | --- |
| **Predefined Function** | **Explanation** |
| =FV (rate, nper, pmt, pv, type) | Calculates the FV of an annuity or annuity due |
| =PV (rate, nper, pmt, fv, type) | Calculates the PV of an annuity or annuity due |
| =RATE (nper, pmt, pv, fv, type) | Interest rate required given a certain number of periods, payment size, and present or future value |
| =NPER (rate, pmt, pv, fv, type) | Number of periods required given a certain interest rate, payment size, and present or future value |
| =EFFECT (nominal\_rate, npery) | Converts an annual percentage rate (APR) into an effective annual rate (EAR) |
| =NOMINAL (effect\_rate, npery) | Converts an effective annual rate (APR) into an annual percentage rate (APR) |
| =PMT (rate, nper, pv, fv, type) | Calculates a blended equal loan payment |
| =IPMT (rate, per, nper, pv, fv, type) | Calculates the interest component of a blended equal loan payment |
| =PPMT (rate, per, nper, pv, fv, type) | Calculates the principal component of blended equal loan payment |